GRIBOV AMBIGUITIES AND HARMONIC ANALYSIS

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Vacuum structure in SU(N) gauge theories is studied both in Coulomb and Landau gauge. It is shown that all Gribov copies are harmonic maps. Implications of harmonicity are discussed. Finally, a new SU(2) Landau gauge copy of the vacuum is presented.

In 1977 Gribov [1,2] observed that the Coulomb (Landau) gauge fixing condition $\partial_i A_i = 0$ ($\partial_\nu A_\nu = 0$) does not uniquely fix the gauge. Later Singer [3] proved that the non-uniqueness of continuous gauge fixing in a compactified domain (S^4) is a characteristic property of all non-abelian gauge theories with a compact gauge group. Even though the existence of gauge fixing degeneracies is widely known, there is no physical interpretation for the degeneracy. Gribov's original idea was to connect the degeneracy with quark confinement but so far this has not been expressed in a satisfactory form. Another possible application would be to formulate in all other continuous gauges, using the degeneracy, a similar vacuum structure we have in the $A_4 = 0$ gauge [4,5]. This idea was first stated by Wadia and Yoneya [6]. In fact this approach seems to be very promising [7–9], at least in the Coulomb gauge. Due to the lack of a formulation of the gauge degeneracy condition in a suitable mathematical framework this approach has not yet been completed.

In this letter I present the vacuum degeneracy problem from a new point of view, and in subsequent papers [10] I will use the presented formalism to give a general approach to the vacuum structure in the widely used Coulomb and Landau gauges. The mathematical framework is harmonic maps between riemannian manifolds, and rather similar ideas have recently been presented in the SU(2) case by Ghika and Visinescu [11]. I will also present a new copy of the vacuum in order to illustrate the vacuum structure in the Coulomb and Landau gauges.

The following notation will be used. At the identity $e \in \mathrm{SU}(N)$ I choose as basis for the su(N) Lie algebra a set of $n = N^2 - 1$ traceless hermitean matrices λ_i, $i = 1, \ldots, n$ such that

$$\mathrm{Tr}\{\lambda_i \lambda_k\} = 2\delta_{ik}. \tag{1}$$

The normalization fixed by the Cartan metric (1) means that for SU(2) the λ_i's are the Pauli matrices and for SU(3) the λ_i's are the Gell-Mann matrices. In the vacuum sector the vector potential A_μ is given by the expression

$$A_\mu = \omega^{-1}\partial_\mu \omega. \tag{2}$$

Here ω is a map from the euclidean space–time R^k ($k = 3$ or 4) to the gauge group SU(N) represented by a unitary matrix. In the Coulomb gauge the summation index ν gets the values $\nu = 1, 2, 3$ and in the Landau gauge $\nu = 1, 2, 3, 4$. With these conventions the divergence condition $\partial_\nu A_\nu = 0$ can be expressed as

$$\partial^2 \omega + (\partial_\nu \omega\, \partial_\nu \omega^{-1})\omega = 0. \tag{3}$$

This can be shown [1] to be the condition that ω is an extremal for the functional

$$E(\omega) = \frac{1}{2} \int_{\mathrm{R}^k} \mathrm{Tr}\{\partial_\nu \omega^{-1} \partial_\nu \omega\}\, \mathrm{d}^k x. \tag{4}$$

Actually the energy functional (4) is ill defined as it diverges. However, we get sense to it by interpreting (4) to be defined on the compact subsets of R^k. This does not alter our conclusions and from a practical point of view we can always use the formal expression (4).

Next I define a positive definite bilinear form ds^2 on the Lie group SU(N) by

$$ds^2 = \mathrm{Tr}\{d\omega^{-1} d\omega\}. \tag{5}$$

Here $d\omega$ is the differential of a given variable unitary matrix ω: $R^k \to SU(N)$. It is easy to check that (5) is bi-invariant, i.e. invariant under left and right action by any constant unitary matrix $\omega_0 \in SU(N)$. In the tangent space at the identity, $T_e\{SU(N)\}$, (5) reduces to the Cartan metric (1). Now one can use a theorem [12] that states that the metric (5) is uniquely given by its value at $T_e\{SU(N)\}$. In fact, (5) is the Killing form on SU(N) which is the unique second rank invariant we have on SU(N). This gives the gauge group SU(N) a riemannian structure.

An energy can be attached for all maps ω from one riemannian manifold $\mathcal{M}$ to another riemannian manifold $\mathcal{N}$ by the following energy functional [13,14]:

$$E(\omega) = \frac{1}{2} \int_{\mathcal{M}} |d\omega|^2 \, d\mu(\mathcal{M}). \tag{6}$$

Here $d\omega$ denotes the differential of ω at the point $p \in \mathcal{M}$. Any deformation of ω which increases the topological irregularity (e.g. by putting in folds and wrinkles) will in general increase the energy $E(\omega)$. One can use the intuitive picture that $\mathcal{M}$ is made of rubber and $\mathcal{N}$ of marble and ω constrains $\mathcal{M}$ to lie on $\mathcal{N}$. The maps ω in some homotopy class $\Pi\{\mathcal{M} : \mathcal{N}\}$ that have minimum energy are the ones that constrain $\mathcal{M}$ to lie on $\mathcal{N}$ in a position of elastic equilibrium. These maps might be expected to have considerable topological regularity. The maps ω that give the minimum energy $E(\omega)$ in some homotopy class are called harmonic maps [13,14]. They are extremals of the energy functional and satisfy the Euler–Lagrange equation

$$\mathrm{div}(d\omega) = 0. \tag{7}$$

A comparison of eqs. (4) and (6) shows that the solutions of (3), which I call copies of the vacuum, are exactly harmonic maps from the riemannian manifold R^k to the riemannian manifold SU(N), when R^k is equipped with its euclidean metric and SU(N) with the Killing metric (5). I will now discuss some properties that are obtained from harmonicity.

(1) We know that both R^k and SU(N) are real analytic manifolds. This implies very strong regularity properties for the harmonic maps ω: $R^k \to SU(N)$: If we require the minimum amount of regularity for admissible ω's (e.g. twice continuously differentiable) in order to assure that we are in the vacuum, which means that the field strength tensor $F_{\mu\nu}$ identically vanishes, the harmonicity implies that the ω's are real analytic [14]. This extraordinary regularity property of harmonic maps shows that we must be very careful when dealing with copies of the vacuum [15] that admit singularities in some subsets of R^k. In what follows I consider only real analytic copies of the vacuum. Due to the strong regularity properties of harmonic maps this will be a very slight restriction.

(2) In the Coulomb gauge the copies of the vacuum cannot essentially depend on time: If they depend on time, the time evolution defines a homotopy in the class of solutions to (7). If the time evolution is essential, it folds and wrinkles the "rubber" R^3 on SU(N). But this changes the tension and that is not possible; thus the time dependence cannot be essential. It can only cause global transformations such as translations and rotations on the manifold SU(N). This means that insofar as the vacuum structure is concerned one can safely set $A_4 = 0$ because the time evolution of $A_4 = \omega^{-1}\partial_4\omega$ is unessential. This enables one to treat the Coulomb gauge vacuum problem as a special case of the $A_4 = 0$ gauge.

(3) On the other hand, in the Landau gauge no vacuum tunneling picture works. This is due to the symmetric introduction of space and time into the gauge condition: If one imposes the Landau gauge condition at a given point in space–time one must know the situation in an infinitesimal neighbourhood of that point in order to assure that the Landau gauge condition is satisfied. This means that no time-slice treatment can occur. Hence there cannot be any tunneling between different vacua. A signal of this mixing of the vacua is the time dependence of the topological flux $q(t)$ defined as [15]

$$q(t_0) = (1/96\pi^2)\epsilon_{ijk}\epsilon_{abc} \int_{t=t_0} A_i^a A_j^b A_k^c \, d^3x. \tag{8}$$

If one calculates this expression for the copy of the vacuum given in ref. [16] one obtains

$$q(t_0) = \pi^{-1} \int_{r=0}^{r=\infty} [\alpha(r, t_0) - \tfrac{1}{2}\sin 2\alpha(r, t_0)], \qquad (9)$$

which varies continuously between 0 and $\frac{1}{2}$. This shows the mixing nature of the Landau gauge: the vacuum structure is there but it is confused by the equal treatment of time and space.

(4) An interesting property of the copies of the vacuum in the SU(2) Coulomb gauge is the following: If one compactifies the spatial part R^3 by imposing the condition

$$\lim_{r\to\infty} \omega(r, \theta, \phi) = \text{constant}, \qquad (10)$$

and assumes real analyticity at the point $\{\infty\}$, the vacuum structure is uniquely given by the trivial vacuum $\omega \equiv$ const. (Note that the real analyticity is an essential but mild assumption, due to the strong regularity properties of harmonic maps; for example, the proof given in ref. [17] fails as it does not exclude the asymptotic behaviour of the form $\epsilon(r) \approx \sin(r^3)/r$, in other words, there are real analytic functions on R^3 that vanish at infinity but the asymptotic behaviour of the derivative can be non-trivial.) The proof goes as follows: First one uses the real analyticity to expand ω in powers of r in the vicinity of $r = \infty$. This gives in first order

$$\omega(r, \theta, \phi) = \omega_0 + (1/r)\omega_1(\theta, \phi) + \ldots, \qquad (11)$$

where ω_0 is a constant matrix. Substituting this into the energy functional (4) one notes that $E(\omega)$ is finite. Now suppose $\omega(x, y, z)$ is a local minimum of the energy functional. This means that $E[\omega(x/a, y/a, z/a)]$ has a vanishing derivative with respect to a at $a = 1$. But $E(\omega_a) = aE(\omega)$, so $E(\omega)$ must vanish. This implies that $\omega(x, y, z)$ is a constant. QED.

One immediately notices that similar results do not necessarily hold in the Landau gauge; simply expand $\omega(r, \theta, \phi, t)$ in the vicinity of $r = \infty$ and notice that $E(\omega)$ may diverge, due to the term $\partial_4 \omega_1^q \partial_4 \omega_1^q$, for example.

(5) In the literature [18] it has been postulated that the Coulomb or the Landau gauge vacuum structure might allow multi-valued copies of the vacuum. But this is in contrast with real analyticity: Multi-valuedness is possible only if one allows singularities in ω.

Next I present a new copy of the vacuum in the SU(2) Landau gauge. The copy arises from the observation that $R^4 = R^2 \times R^2$. This splitting of R^4 means that if there is a copy of the vacuum in R^2 satisfying the two-dimensional Landau gauge condition, one gets a copy in R^4 by multiplying the two copies of the R^2 parts. For this purpose I will study the vacuum structure in R^2.

A separable copy of the vacuum, $\omega(x, y)$, can be written in the form

$$\omega(r, \phi) = \exp[i\alpha(r)\hat{\boldsymbol{n}}(\phi)\cdot\hat{\boldsymbol{\sigma}}], \qquad (12)$$

where $\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{n}} = 1$. Eq. (12) satisfies the Landau gauge condition if and only if $\alpha(r)$ and $\hat{\boldsymbol{n}}(\phi)$ satisfy [16]

$$\partial^2\alpha = -\tfrac{1}{2}\sin(2\alpha)\, \hat{\boldsymbol{n}}\cdot\partial^2\hat{\boldsymbol{n}}, \qquad (13)$$

$$\partial^2\hat{\boldsymbol{n}} \sin\alpha + \hat{\boldsymbol{n}}\, \partial^2\alpha \cos\alpha = (\hat{\boldsymbol{n}}\cdot\partial^2\hat{\boldsymbol{n}})\hat{\boldsymbol{n}} \sin^3\alpha. \qquad (14)$$

Separability implies that (14) is satisfied if (13) holds. From (13) follows

$$L^2\hat{\boldsymbol{n}} = \lambda^2\hat{\boldsymbol{n}}, \qquad (15)$$

$$\partial^2\alpha = \tfrac{1}{2}\lambda^2 r^{-2}\sin 2\alpha, \qquad (16)$$

where λ is a separation constant. The solution $\alpha(r)$ of (16) behaves like $r^{|\lambda|}$ near $r = 0$. Since $\alpha(r)$ is real analytic one must set λ = integer. The general solution of (15) is given by

$$\hat{\boldsymbol{n}}(\phi) = \hat{\mathbf{A}}\begin{pmatrix}\sin\lambda\phi \\ \cos\lambda\phi \\ 0\end{pmatrix}, \qquad (17)$$

where $\hat{\mathbf{A}}$ is a constant orthogonal matrix. Substituting $t = \lambda \ln(r)$ into (16) one gets the pendulum equation

$$\ddot{\alpha} = \tfrac{1}{2}\sin 2\alpha. \qquad (18)$$

The exact solution to this equation is

$$\alpha = 2\arctan[(r/r_0)^{|\lambda|}]. \qquad (19)$$

Note that as $r \to \infty$, ω has the following limit:

$$\lim_{r\to\infty} \omega(r, \phi) = \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix}. \qquad (20)$$

This allows one to compactify $R^2 \cup \{\infty\} \sim S^2$. If one now multiplies two such solutions, $\omega_1(x, y)$ and $\omega_2(z, t)$ for example, one gets a family of copies of the vacuum satisfying the Landau gauge condition. This family can be classified by $Z \times Z = (\lambda_1, \lambda_2)$, and has the interesting property that it lies in the compact space $S^2 \times S^2$. An interesting observation is that $\omega_1(x, y)$ itself gives a copy of the vacuum satisfying both the Coulomb and the Landau gauge fixing conditions. This shows that in the $q = 0$ [eq. (8)] sector there are non-trivial configurations besides the $A_\mu \equiv 0$ configuration.

As the last remark I note that the two-dimensional copies presented here can be used with the three-dimensional copies of the vacuum [1,2] to yield copies of the vacuum in all dimensions $k \geqslant 2$.

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